UDC 531.38

ON A FAMILY OF PERIODIC MOTIONS OF A HEAVY SOLID WITH A FIXED POINT

PMM Vol. 41, № 3, 1977, pp. 553-556
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(Received February 19, 1976)

The equation of motion of a heavy solid with a fixed point is transformed, using isothermal coordinates on the inertia ellipsoid, to the equations of plane motion of a fictitious material point. The Poincaré method of small parameter is used to prove the existence of a new family of periodic solutions of the problem in question. It is assumed that the solid differs little from a solid with dynamic axial symmetry.

Let us consider the motion of a heavy solid about a fixed point. We denote by p, q, and r the projections of the angular velocity of the body on the principal axes of inertia; α, β, and γ are the direction cosines of the vertical; A, B and C are the principal moments of inertia; a, b, and c are the coordinates of the center of gravity in the moving coordinate system. The Lagrangian

$$L = {}^1/_2 (Ap^2 + Bq^2 + Cr^2) - mg(a\alpha + b\beta + c\gamma) \tag{1}$$

of the problem does not contain the angle of precession ψ, and this makes it possible to transform the equation of motion using the Routh method and a cyclic integral

$$\partial L / \partial \psi^{\cdot} = A\alpha(\psi^{\cdot}\alpha + \theta^{\cdot}\cos\varphi) + B\beta(\psi^{\cdot}\beta - \theta^{\cdot}\sin\varphi) + C\gamma(\psi^{\cdot}\gamma + \varphi^{\cdot}) = f$$

Here f is an arbitrary constant, φ and θ are the angles of self-rotation and nutation, respectively.

The Routh function can be written in the present case in the form [1]

$$R = R_2 + R_1 + R_0$$

$$R_2 = \frac{1}{2} D \Big\{ \varphi^{\cdot 2} C (A\alpha^2 + B\beta^2) - 2\varphi^{\cdot}\theta^{\cdot} C\gamma (A\alpha\cos\varphi - B\beta\sin\varphi) +$$

$$\theta^{\cdot 2}\Big[\frac{1}{D}(A\cos^2\varphi + B\sin^2\varphi) - (A\alpha\cos\varphi - B\beta\sin\varphi)^2\Big]\Big\}$$

$$R_1 = fD[\varphi^{\cdot} C\gamma + \theta^{\cdot}(A\alpha\cos\varphi - B\beta\sin\varphi)]$$

$$R_0 = -mg(a\alpha + b\beta + c\gamma) - {}^1/_2 f^2 D$$

where $\sqrt{D}$ denotes the disturbance from the fixed point to the tangent plane of the inertia ellipsoid

$$D^{-1} = A\alpha^2 + B\beta^2 + C\gamma^2$$

Next we shall carry out two coordinate transformations, one after the other. First we introduce the isothermal coordinates u and s on the surface of the inertia ellipsoid using the formulas

$$\varphi = \mathrm{Arctg}\,\frac{\mathrm{sn}\,u}{\mathrm{cn}\,u\,\mathrm{sn}\,s}, \quad \theta = \arccos\frac{\mathrm{dn}\,u\,\mathrm{cn}\,s}{\mathrm{dn}\,s}$$

in which the modulus of the elliptical functions of the argument u is $k = [(A-B)C\,/\,(A-C)B]^{1/2}$ and the elliptical functions of the argument s depend on the auxilliary modulus $k' = (1-k^2)^{1/2}$. The next coordinate transformation is given by the relations

$$x = \int_0^u \mu_1(u)\,du, \quad y = -\int_0^s \mu_2(s)\,ds$$

$$\mu_1^2(u) = A\,\mathrm{sn}^2 u + B\,\mathrm{cn}^2 u, \quad \mu_2^2(s) = \frac{1}{\mathrm{dn}^2 s}\,(A - Bk'^2\,\mathrm{sn}^2 s)$$

The Routh function can be written in the new variables in the form

$$R = \frac{1}{2} I\,(x^{\cdot 2} + y^{\cdot 2}) + \frac{f}{\Lambda M}\,\frac{\nu_1(u)\,\lambda_2(s)}{\mu_1(u)}\,x^{\cdot} + \frac{f}{\Lambda M}\,\frac{\nu_2(s)\,\lambda_1(u)}{\mu_2(s)}\,y^{\cdot}$$

$$I = \frac{Ck}{\Lambda}\left(k'^2\frac{\mathrm{sn}^2 s}{\mathrm{dn}^2 s} + \mathrm{cn}^2 u\right), \quad M = 1 - \mathrm{cn}^2 u\,\mathrm{cn}^2 s$$

$$\Lambda = \frac{1}{\mathrm{dn}^2 s}\,(Ak^2\,\mathrm{sn}^2 u + Bk^2\,\mathrm{cn}^2 u\,\mathrm{sn}^2 s + C\,\mathrm{dn}^2 u\,\mathrm{cn}^2 s)$$

$$\nu_1(u) = C\,\mathrm{dn}^2 u + k^2\,(A - B)\,\mathrm{sn}^2 u\,\mathrm{cn}^2 u$$

$$\nu_2(s) = C\,\mathrm{cn}^2 s - k^2\,(A - B)\,\frac{\mathrm{sn}^2 s}{\mathrm{dn}^2 s}$$

$$\lambda_1(u) = \mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u, \quad \lambda_2(s) = \mathrm{sn}\,s\,\mathrm{cn}\,s\,/\,\mathrm{dn}\,s$$

Next we transform the equations of motion

$$\frac{d}{dt}\left(\frac{\partial R}{\partial x^{\cdot}}\right) - \frac{\partial R}{\partial x} = \frac{\partial U}{\partial x}, \quad \frac{d}{dt}\left(\frac{\partial R}{\partial y^{\cdot}}\right) - \frac{\partial R}{\partial y} = \frac{\partial U}{\partial y}$$

taking into account the kinetic energy integral

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = U + h$$

in which the function U is defined by the second term of (1) written in the new variables, and h is a constant.

Introducing now a new regularizing variable with the help of the relation $dt = I d\tau$, we obtain the following system of equations of motion

$$x'' - f\Omega y' = \frac{\partial V}{\partial x}, \quad y'' + f\Omega x' = \frac{\partial V}{\partial y} \tag{2}$$

$$\Omega = \frac{1}{\mu_1(u)\,\mu_2(s)}\left\{\nu_2(s)\,\frac{\partial}{\partial u}\left[\frac{\lambda_1(u)}{\Lambda M}\right] + \nu_1(u)\,\frac{\partial}{\partial s}\left[\frac{\lambda_2(s)}{\Lambda M}\right]\right\}$$

$$V = \frac{I}{k}\left[h - \frac{mg}{\mathrm{dn}\,s}\,(ak\,\mathrm{sn}\,u + bk\,\mathrm{cn}\,u\,\mathrm{sn}\,s + c\,\mathrm{dn}\,u\,\mathrm{cn}\,s) - \frac{f^2}{2\Lambda}\right]$$

where prime denotes differentiation with respect to τ.

In this manner we have reduced the equations of motion of a solid to a fourth order system describing the motion of a fictitious material point in a plane, under the action of potential and gyroscopic forces.

The transformed system admits the Jacobi integral

$$x'^2 + y'^2 = 2V \tag{3}$$

Next we prove the existence of and construct the periodic solutions of the system (2), using the Poicaré's small parameter method. Expanding the function Ω into trigonometric series we confirm that it is of the order of k^2. Assuming the quantity f to be arbitrary and $\Omega = k^2\Omega^*$, we can use the Poincaré's method to establish the existence of a new family of periodic solutions of the problem using the modulus k as the small parameter [2].

Setting $k = 0$ we obtain from (2) the following simplified system

$$x_0'' = \frac{\partial V_0}{\partial x_0}, \quad y_0'' = \frac{\partial V_0}{\partial y_0} \tag{4}$$

$$V_0 = (h - mgc)\left(\cos^2 \frac{x_0}{\sqrt{A}} + \operatorname{sh}^2 \frac{y_0}{\sqrt{A}}\right)$$

Multiplying both parts of (4) by x_0' and y_0' respectively, and integrating, we obtain

$$x_0'^2 = 2(h - mgc)\cos^2 \frac{x_0}{\sqrt{A}} + C_1$$

$$y_0'^2 = 2(h - mgc)\operatorname{sh}^2 \frac{y_0}{\sqrt{A}} + C_2$$

Here $C_1 + C_2 = 0$, which follows from the Jacobi integral (3).

If the arbitrary constants h and C_1 are restricted by the conditions $h - mgc > 0$, and $C_1 < 0$, the system (4) admits the following solution

$$\sin \frac{x_0}{\sqrt{A}} = \varkappa \operatorname{sn}(w_1, \varkappa), \quad \operatorname{sh} \frac{y_0}{\sqrt{A}} = \varkappa' \frac{\operatorname{sn}(w_2, \varkappa)}{\operatorname{cn}(w_2, \varkappa)}$$

$$\varkappa^2 = (A\sigma^2 - C_1) / A\sigma^2, \quad \sigma = [2(h - mgc) / A]^{1/2}$$

$$w_i = \sigma(\tau - \tau_i), \quad i = 1, 2$$

with the period $T = 4\sigma^{-1}K(\varkappa)$.

The system of differential equations in its first order approximation

$$x_1'' - \frac{2}{A}(h - mgc)\left(\sin^2 \frac{x_0}{\sqrt{A}} - \cos^2 \frac{x_0}{\sqrt{A}}\right)x_1 = f\Omega_0^* y_0'$$

$$y_1'' - \frac{2}{A}(h - mgc)\left(\operatorname{sh}^2 \frac{y_0}{\sqrt{A}} + \operatorname{ch}^2 \frac{y_0}{\sqrt{A}}\right)y_1 = -f\Omega_0^* x_0'$$

where Ω_0^* is the value of the function Ω^* at $k = 0$ has the solution

$$x_1 = x_0' \left\{\int \frac{\beta_1}{x_0'^2}\, d\tau + \beta_2 + \int \frac{f}{x_0'^2}\, \omega\, d\tau\right\}$$

$$y_1 = y_0' \left\{ \int \frac{\beta_3}{y_0'^2} d\tau + \beta_4 + \int \frac{f}{y_0'^2} \omega\, d\tau \right\}$$

$$x_0' = \sqrt{A}\,\sigma\varkappa \operatorname{cn}(w_1, \varkappa), \quad y_0' = \frac{\sqrt{A}\,\sigma\varkappa}{\operatorname{cn}(w_2, \varkappa)}, \quad \omega = \int \Omega_0^* x_0' y_0'\, d\tau$$

where β_i are arbitrary constants.

Since the system (2) is invariant under the substitution $\tau \to -\tau$, $x \to x$, $y \to -y$, $x^\cdot \to -x^\cdot$, $y^\cdot \to y^\cdot$, the conditions of the periodicity of the solution can be written, in accordance with the symmetry theorem [3], as follows:

$$\psi_1 = x^\cdot(0) = 0, \quad \psi_2 = x^\cdot(T/2) = 0$$
$$\psi_3 = y(0) = 0, \quad \psi_4 = y(T/2) = 0 \tag{5}$$

It can be shown that conditions (5) will be satisfied if $J_{12} \neq 0$, $J_{34} \neq 0$ and

$$J_{mn} = \left[\frac{D(\psi_m, \psi_n)}{D(\beta_m, \beta_n)}\right]_{k=0}$$

Equations (5) can be written in the explicit form as follows:

$$x_1^\cdot(0) = \frac{\beta_1}{\sqrt{A}\,\varkappa\varkappa'^2\sigma}\{\varkappa' M(\varkappa, \varkappa') + \varkappa'^2 - \varkappa^2\} - \beta_2 \sqrt{A}\,\varkappa\varkappa'\sigma^2 = 0$$

$$x_1^\cdot\left(\frac{T}{2}\right) = \frac{\beta_1}{\sqrt{A}\,\varkappa\varkappa'^2\sigma}\{3\varkappa' M(\varkappa, \varkappa') - 1\} + \beta_2 \sqrt{A}\,\varkappa\varkappa'\sigma^2 = 0$$

$$y_1(0) = \sqrt{A}\,\beta_4 \varkappa' \sigma = 0$$

$$y_1\left(\frac{T}{2}\right) = -\frac{2\beta_3}{\sqrt{A}\,\sigma^2\varkappa^2\varkappa'} M(\varkappa, \varkappa') - \sqrt{A}\,\beta_4 \sigma\varkappa' = 0$$

$$M(\varkappa, \varkappa') = E(\varkappa) - \varkappa'^2 K(\varkappa)$$

and the latter relations yield

$$J_{12} = 2\sigma[\varkappa' - M(\varkappa, \varkappa')] \neq 0,$$

$$J_{34} = \frac{2}{\sigma\varkappa^2} M(\varkappa, \varkappa') \neq 0$$

This means that the system (2) admits the new family of periodic solutions of the problem in question.

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Translated by L. K.